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FREQUENCY LOCKING IN TISSULAR COUPLING

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Abstract. We expose a framework, inspired by biological observations, dedicated to modeling complex living systems as coupled systems. In particular, we use this framework to adress a main question in the field of living systems: the synchronization phenomenon. This kind of model, named tissular coupling, is quite general and, using different methods from those usually used in this field of research, we reach global results relative to the frequencies locking problem in both finite and continuous populations.

Keywords. Differential systems, finite coupled systems, infinite coupled systems, synchronization, frequency locking.

1 Introduction

Synchronization is an extremely important and interesting emergent property of complex systems. The first example found in literature goes back to the 17th century with Christiaan Huygens' clocks [10, 1]. Although this example rose from artificial systems, this kind of emergent behavior can be found in natural systems at any scale (from cell to whole ecological systems). Indeed, biology abounds with periodic and synchronized phenomena and Ilya Prigogine's work gave a first general explanation to this matter: such behaviors arise from a dissipative structure generally associated to a nonlinear dynamics [19]. Biological systems are open, they evolve far from thermodynamic equilibrium and are subject to numerous regulating processes, leading to highly nonlinear dynamics. Therefore, periodic behaviors appear (with or without synchronization) at any scale [20]. More generally, life itself is governed by circadian rhythms [8]. Those phenomena are as much attractive as they are often spectacular: from cicada populations that appear spontaneously every ten or thirteen years [9] or networks of heart cells that beat together [16] to huge swarms in which fireflies, gathered in a same tree, flash simultaneously [2]. This synchronization phenomenon occupies a privileged position among emergent collective phenomena and more generally in the field of complex

systems for it has various applications in neuroscience, ecology, earth Science, for instance [24, 23, 15], as well as in the field of coupled dynamical systems, especially through the notion of chaotic systems' synchronization [17, 6] and the study of coupled-oscillators [12]. This wide source of examples leads the field of research to be highly interdisciplinary, from pure theory to concrete applications and experimentations.

In the second section of our work, and before adressing a specific question related to synchronization, we build a modeling framework inspired by biological systems in order to handle complex systems. This framework, the tissular coupling, is based on a quite general vision of complex systems: they are constituted of interacting subsystems, named cells. A tissular coupling is a quite general object as the number of cell is not necessarily finished, for the population $\mathcal{P}$ (the set of cell) may be a continuous set. Moreover, the cell may have different behavior, as each of them own a proper dynamical system (which is not supposed to be an oscillator). Another important characteristic is that those dynamical systems undergo interactions among the whole population and those interactions are not limited to some simple cases usualy handled in litterature. Thus, tissular coupling is a very general model of complex systems that does not focuse solely on isolated dynamics, nor on interactions between subsystems, but that allows a balance of these two main characteristic of complex and/or living systems.

Once this framework exposed, in the last part of this section, we start our way to adress synchronization issues. The classical concept of synchronization is related to the locking of the basic frequencies and instantaneous phases of regular oscillations. One of the most successful attempts to explore this emergent property is due to Kuramoto [13, 14]. As in Kuramoto's work, those questions are usually addressed by studying specific kinds of coupled systems (see for instance [4, 21, 7]). Using all the classical methods available in the field of dynamical systems, researchers study specific trajectories of those systems in order to get information on possible attracting synchronized state [25, 12, 21, 18, 7, 11]. Most of these works always deal with a specific way of coupling dynamical systems, which we call "coupling in the final space": one adds a quantity that models interactions to the derivatives of the systems. This leads to equations with the following typical shape (here, there are only two coupled systems):

$$x_1'(t) = F\Big(x_1(t)\Big) + \varepsilon G_1\big(x_1(t), x_2(t)\big)$$
$$x_2'(t) = F\Big(x_2(t)\Big) + \varepsilon G_2\big(x_1(t), x_2(t)\big)$$

The problem is restated in terms of phase-shift variables and efforts are made to detect stable states and to prove their stability.

Our approach is quite innovative as we explore synchronization starting from another point of view: we focus on the coupler rather than on the

dynamical systems to be coupled. First, contrary to what precedes, in this work we study exclusively the "coupling in the initial space", which means that the coupling quantity applies inside the map F, which leads us to the following type of equation:

$$x_1'(t) = F\Big(x_1(t) + \varepsilon G_1(x_1(t), x_2(t))\Big)$$
$$x_2'(t) = F\Big(x_2(t) + \varepsilon G_2(x_1(t), x_2(t))\Big)$$

This kind of coupled systems is sometimes studied (for instance in [11]) but never broadly (indeed, if one wants some quantitative results about convergence of trajectories, one must work with specific equations and dynamical systems). We show that this latter shape is still very general. Moreover, under a weak assumption on the dynamical systems to be coupled (named periodically-injectivity) it allows us to considerably reduce our synchronization issues to a structural problem which becomes independant from the own dynamic of each cell. This new problem depends actually only on the coupler, the interactions between cells.

Then, at last we go to the heart of the problem and study synchronization issues. Convinced that this phenomenon is completely natural in a large variety of coupled dynamical systems, we propose a new approach to the subject: rather than trying to prove that synchronization actually takes place, we are able, in this context, to search conditions under which frequencies are locked as soon as the whole system oscillates. We stress the point that we do not assume that the dynamical systems owned by cells are oscillators, but only that each cell has a periodic trajectory (which is completely different, and is a really much weaker assumption). We give answers to this question in sections 3 and 4 in which we exhibit some natural conditions under which we are able to prove the main results of this paper: cases of frequencies locking in an finite population and in a infinite compact and connected one. In other words, our results show that the following alternative naturaly raises in many cases: either the whole population is synchronized, or its cells can't all have periodic behaviors.

2 Basic material and notation

As our model is inspired by cellular tissues, several terms clearly come from the vocabulary used to describe those kinds of complex systems. Nevertheless, its scope is not limited to cells nor cell tissues. Moreover, in this first work we only deal with smooth mathematical objects, but it's naturally possible to extend this work to non continuous trajectories (to handle pulse oscillators for example). This is actually a piece of work we are achieving.

2.1 Model of population behavior

Here are the basic compounds of our model (for the sake of concision and in order to simplify the writing, notations differ from the usual ones used in the introduction):

- a **population** $\mathcal{P}$ is a measured space with finite measure. Typical populations are finite sets with counting measure or measurable subsets of $\mathbb{R}^m$ with Lebesgue measure. Every $p \in \mathcal{P}$ is called a cell. In most papers (see for instance [12]) this population is implicitly defined and has only two cells (sometimes a finite number N), each cell being associated to an oscillator. Even when the population is a continuum, trajectories are supposed to be regular along the population, not only along time (see for instance [4]). We believe that the population of systems to be coupled is at least as much important as the individual systems themselves. Moreover it seems obvious that even if the population is continuous, states along cells do not need to be regular (see our discussion in [5]);

- for the sake of concision, we suppose that the systems we want to study are $\mathbb{R}^n$-valued. Thus, a **state** of $\mathcal{P}$ is an element of $\mathcal{S} = \mathcal{M}_b(\mathcal{P}, \mathbb{R}^n)$, the space of measurable bounded applications from $\mathcal{P}$ to $\mathbb{R}^n$. A state of $\mathcal{P}$ will be denoted by s, $s(p)$ standing for the state of the cell p;

- let $I \subset \mathbb{R}$ be an interval and $r \in \mathbb{N}$. A **trajectory** of $\mathcal{P}$ is an element of $\mathcal{F}(I, \mathcal{S})$. For the sake of concision, any trajectory will be written s again, and $s(t, p)$ stands for the state of the cell p at time t (which should be written $s(t)(p)$). Then, we define the space of trajectories as:

$$\mathcal{T}^r = \left\{ \mathcal{C}^r(I, \mathcal{S}),\ \forall J \Subset I,\ \sup_{J \times \mathcal{P}} \|s(t,p)\| < \infty \right\}$$

  Again, we believe that in order to understand complex systems, we must be able to handle both individual components and global population, this is why we can't restrict our study to the cells' behaviors and we handle the population's trajectory as a whole.
  A trajectory s is regular only along the time t (except in terms of measurability), thus we use the following unambiguous notation:

$$s'(t,p) = \partial_1 s(t,p)$$

- as there is no ambiguity, all norms will be denoted in the same manner:

$$\begin{aligned} s \in \mathcal{S}: \ \|s\| &= \sup_{\mathcal{P}} \|s(p)\| \\ s \in \mathcal{T}: \ \|s\| &= \sup_{I\times\mathcal{P}} \|s(p,t)\| \\ \|s(t,.)\| &= \sup_{\mathcal{P}} \|s(t,p)\| \\ \|s(.,p)\| &= \sup_{I} \|s(t,p)\| \end{aligned}$$

Equipped with this kind of norm, $\mathcal{T}^0$ becomes obviously a Banach space (this norm is adapted to smooth problems, when we will deal with non-regular trajectories we will obviously need other classical norms).

2.2 Periodic motions and properties of periods on $\mathcal{P}$

Now, we define the notion of periodic trajectories on a population and some mathematical tools related to their study. On the contrary of what most studies on synchronization issues state, we do not suppose that the cells are oscillators, we will only assume they exhibit periodic behaviors (the first assumption implies the second, but the opposite is clearly false). Thus, we need to define periodic trajectories of a whole population:

Definition 2.1. *A **period on** $\mathcal{P}$ is a map τ from $\mathcal{P}$ to $\mathbb{R}_+^*$. A trajectory $s \in \mathcal{T}^r$ is said to be τ-**periodic** if for any $p \in \mathcal{P}$, $s(.,p)$ is $\tau(p)$-periodic. $\tau(p)$ is then called the **period of** p. The space of such trajectories is written $\mathcal{T}_\tau^r$.*

If $s \in \mathcal{T}_\tau^1$, for any p the map $s(.,p)$ equals its Fourier's serie. We write:

$$e_\tau^k(t,p) = \exp\left(\frac{2i\pi kt}{\tau(p)}\right)$$

so that we have:

$$s(t,p) = \sum_{k\in\mathbb{Z}} \widehat{s}(k,p) e_\tau^k(t,p)$$

with normal convergence, and obviously:

$$\widehat{s}(k,p) = \frac{1}{\tau(p)} \int_0^{\tau(p)} s(t,p) e_\tau^k(t,p) dt$$

Note that these quantities are $\mathbb{R}^n$-valued.

Here follows a corollary of the ergodic theorem [5] (and for references to ergodic theory see for instance [22, 3]):

Proposition 2.1 (Ergodic alternative)**.** *Let f be a smooth real-valued map, τ-periodic an $\tau' > 0$. We define the τ'-average of f:*

$$a_{\tau'}(f)(t) = \lim_{N \to +\infty} \frac{1}{N+1} \sum_{k=0}^{N} f(t + k\tau')$$

Then the following alternative holds:

- *τ and τ' are $\mathbb{Z}$-free (non commensurable), then $a_{\tau'}(f)$ is constant and equals $\tau \widehat{f}(0)$;*
- *there's an irreducible fraction $\frac{a}{b}$ such that $b\tau = a\tau'$, then $a_{\tau'}(f)$ is $\frac{\tau'}{b}$-periodic.*

In order to prove synchronization, the first step consists in showing that periods are commensurable. We need the following notion: a (finite) subset $\{\tau_1, \ldots, \tau_k\}$ of $\mathbb{R}$ is said to be **dependent** (over $\mathbb{Z}$) if there exists some integers $l_1, \ldots, l_k$ not all null and such that:

$$l_1\tau_1 + \ldots + l_k\tau_k = 0$$

A way to handle some properties of a period τ is to consider some partitions of $\mathcal{P}$. Let $p \in \mathcal{P}$ and $a \in \mathbb{R}^*$, we define the following application:

$$\tau_a(p) = a\tau(p)$$

and for any subset $A \subset \mathbb{R}$:

$$\mathcal{P}_p^A = \bigcup_{a \in A} \tau_a^{-1}(\tau(p))$$

In other words:

$$\mathcal{P}_p^A = \{q \in \mathcal{P} \ : \ \exists a \in A,\, a\tau(q) = \tau(p)\}$$

As soon as $1 \in A$, the set of all distinct $\mathcal{P}_p^A$ is a partition of $\mathcal{P}$, we write $\mathcal{P}^A$ this partition (with non-empty sets). Main examples are:

- if all the periods constitute a dependent set, then $\mathcal{P}^{\mathbb{Q}} = \{\mathcal{P}\}$;
- if all the periods are identical ($\mathcal{P}$ is synchronized), then $\mathcal{P}^1 = \{\mathcal{P}\}$.

Let's consider differential systems on $\mathcal{P}$ now to complete our description of their coupling.

2.3 Differential systems on $\mathcal{P}$

As we want to reach the most general result, each cell p is supposed to behave according to an autonomous differential system given by an application F_p. Non-autonomous differential systems could be assimilated to the case of synchronization with the help of an external force. Moreover, as we work with any kind of population, this external force could be assimilated to new cells and integrated in the population. For the sake of concision, we will assume that F_p is smooth. Then, we have a family of applications $\{F_p\}_{p\in\mathcal{P}}$ and we define the extension $F_{\mathcal{P}}$ of this family to $\mathcal{S}$:

$$\begin{array}{rccc} F_{\mathcal{P}}: & \mathcal{S} & \longrightarrow & \mathcal{S} \\ & s & \longmapsto & F_{\mathcal{P}}(s) \end{array}$$

with

$$F_{\mathcal{P}}(s)(p) = F_p(s(p))$$

$F_{\mathcal{P}}$ is said to be $\mathcal{C}^r$ if every F_p is $\mathcal{C}^r$ and if for any bounded subset B of $\mathbb{R}^n$ we have:

$$\sup_{x\in B\times p\in\mathcal{P}} \|dF_p(x)\| < \infty$$

It is important to notice that we won't have to specify the latter maps, not even to suppose that they are oscillators. In order to prove our results we will only need to assume that they are periodic-injective, which is a weak assumption (see section 2.5).

Coupler on $\mathcal{P}$ Now, we consider how to couple the differential systems given by $F_{\mathcal{P}}$. Usually [18], the coupling is made by adding to $s'(t,p)$ a quantity that depends on $s(t,.)$:

$$s'(t,p) = F_p(s(t,p)) + c(s(t,.))(p) \tag{1}$$

where $c(s(t,.))$, an element of $\mathcal{S}$, is a function of the global state of $\mathcal{P}$. This is what we naturally name a coupling **in the final space**. Another way of coupling is **in the initial space**:

$$s'(t,p) = F_{\mathcal{P}}\Big(c\big(s(t,.)\big)\Big)(p)$$

Coupling in the final space or in the initial space may have different interpretations. The first coupling method could be seen as physical exchanges and the second one as an instantaneous exchanges (for instance an high speed information exchange which could be modeled as instantaneous with regards to the system time scale). But, in most situations, those two couplings are equivalent. Indeed, the right-hand-side term of the equation (1) has the following shape:

$$F(x) + \varepsilon$$

so that, as soon as this r.h.s. stays in the range of the map F, one can find a vector y:

$$F(x) + \varepsilon = F(y)$$

writing $\delta = y - x$ we have:

$$F(x) + \varepsilon = F(x + \delta)$$

In fact, this case is quite general as ε is usually small, and even if it is not, we will study only periodical trajectories occurring essentially around limit cycles surrounded by the range of F. For these reasons, we will concentrate on the last kind of coupling. Now we give technical details about our method of coupling, by the way of a **coupler**:

$$\begin{array}{rccc} C: & \mathcal{S} & \longrightarrow & \mathcal{S} \\ & s & \longmapsto & C(s) \end{array}$$

This coupler is said to be **linear** if there is an application c satisfying:

$$\begin{array}{rccc} c: & \mathbb{R}^n \times \mathcal{P} & \longrightarrow & \mathcal{L}\left(\mathcal{S}, \mathbb{R}^n\right) \\ & (x, p) & \longmapsto & c(x, p) \end{array}$$

such that for all $p \in \mathcal{P}$:

$$C(s)(p) = c\left(s(p), p\right).s$$

In other words, depending on the cell p (*e.g.* its spatial position) and on its state $s(p)$, the coupling compute linearly (as a mean) the pseudo-state $c\left(s(p), p\right).s$ dictating the evolution of this cell, instead of the state $s(p)$ which is used if the cell is isolated (without coupling).
Sometimes the following point of view, the operator one, is be useful:

$$\begin{array}{rccc} L_c: & \mathcal{S} & \longrightarrow & \mathcal{L}(\mathcal{S}, \mathcal{S}) \\ & s & \longmapsto & L_c(s) \end{array}$$

where:

$$[L_c(s).s](p) = c\left(s(p), p\right).s$$

If L_c is constant, the coupler is said to be **uniformly linear** and is naturally associated to an element of $\mathcal{L}(\mathcal{S}, \mathcal{S})$. The case of uniformly linear coupler is the simpler one: the way a cell is influenced by its environment does not depends on its proper state. It is an approximation of what happens in real systems, but this simplification allows us to reach very global results.

To get familiar with the notion of coupler, let's write out how it transforms a trajectory of $\mathcal{P}$. If $s \in \mathcal{T}$, the effect of a coupler on this trajectory is:

$$C(s)(t, p) = c(s(t, p), p).s(t, .)$$

2.4 Tissular coupling

We are now able to define a tissular coupler. Given a population $\mathcal{P}$, a coupler C is said to be a **tissular coupler** if the associated application c satisfies:

$$c(x,p).s = \int_{\mathcal{P}} c_d(x,p,q).s(q)dq + c_a(x,p).s(p)$$

where c_d and c_a are continuous and are respectively called diffuse part and atomic part of c (and of C). The diffuse part represents the influence of the environment (the whole population) on the behavior of the cell p. The atomic part models the self-influence of p. We could have merged this into a unique linear operator, containing a Dirac impulse, but for both comprehension and manipulation purposes we think that this shape is better and less theoretical.

Finally, we can define a tissular coupling by specifying the equation that any solution should satisfy. Given a population $\mathcal{P}$, a family of applications $\{F_p\}_{\mathcal{P}}$ and a tissular coupler C, the **tissular coupling** $(\mathcal{P}, F_{\mathcal{P}}, C)$ is defined by the equation:

$$s' = F_{\mathcal{P}} \circ C(s) \tag{2}$$

in other words, any solution s satisfies for all $(t,p) \in I \times \mathcal{P}$:

$$s'(t,p) = F_p\left(\int_{\mathcal{P}} c_d(s(t,p),p,q).s(t,q)dq + c_a(s(t,p),p).s(t,p)\right)$$

We naturally end this introduction of the tissular coupling with a result on the existence of solutions, which is proved in a classical way [5]:

Theorem 2.1. *Suppose $F_{\mathcal{P}}$ is $\mathcal{C}^1$ and c_d and c_a are locally lipschtizian in their first variable. Given any initial condition (t_0, s_0) in $I \times \mathcal{S}$, the tissular coupling given by equation (2) admits a unique maximal solution.*

We can then go further and begin to work on the heart of our matter.

2.5 Last step: Problem Reduction

Our first step in reducing the problem was to handle coupling at the source. Now, with some natural assumption made on $F_{\mathcal{P}}$ we will reduce the problem to a structural one.

Proposition 2.2. *Let $(\mathcal{P}, C, F_{\mathcal{P}})$ be a tissular coupling and τ a period on $\mathcal{P}$. Let's assume that there exists a family $\{U_p\}_{p\in\mathcal{P}}$ of open subsets of $\mathbb{R}^n$ such that for all $p \in \mathcal{P}$ F_p is injective on U_p. If s is a τ-periodic solution of the coupling such that for all $p \in \mathcal{P}$, $s(I,p) \subset U_p$, then s belongs to the following set:*

$$A(\tau, C) = \mathcal{T}^\infty \cup C^{-1}(\mathcal{T}^\infty)$$

Thus, we have drastically reduced the problem: under the assumptions of the proposition 2.2 we can get rid of $F_{\mathcal{P}}$ and only work with the coupler C and the space of periodic solutions.
The thrid section exhibits the main results of our work on finite populations, as the fourth one will concern continuous population.

3 Synchronization of finite population

Here we focus on a coupler in the case of a finite population $\mathcal{P}$:

$$\mathcal{P} = \{p_1, \ldots, p_k\}$$

If C is a tissular coupling, the diffuse part is sufficient to define it, thus we can also write it for $x \in \mathbb{R}^n$ as:

$$c_d(x, p_i, pj) = c_{ij}(x) \in \mathcal{L}\left(R^n\right)$$

L_c adopts naturally a matrix shape:

$$L_c(x) = \begin{bmatrix} c_{11}(x) & \ldots & c_{1k}(x) \\ \vdots & \ddots & \vdots \\ c_{k1}(x) & \ldots & c_{kk}(x) \end{bmatrix}$$

We point out that each $c_{ij}(x)$ is a linear application, an element of $\mathcal{L}\left(\mathbb{R}^n, \mathbb{R}^n\right)$ (to be precise and to anticipate on some theoretical generalizations of the tissular coupling, we note that $L_c(x)$ is the matrix of a linear application on $(\mathbb{R}^n)^k$ seen as a module on the ring $\mathcal{L}\left(\mathbb{R}^n, \mathbb{R}^n\right)$).
The image of the motion $s(., p_i)$ is thus given by:

$$C(s)(t, p_i) = \sum_{j=1}^{k} c_{ij}(s(t, p_i))s(t, p_j)$$

The kind of assumptions we are about to make on C involve linear properties. Especially we work with the rank of L_c and of sub-matrices. Thus, if $J \subset \{1, \cdots, k\}$ contains l elements, we write $I^c = \{1, \cdots, k\} - J$. If $M = (m_{ij})$ is an $k \times k$ matrix, we define M^I as the $l \times (l - k)$ matrix which coefficients are the $m_{i,j}$ for $(i, j) \in I \times I^c$. For $s \in \mathcal{S}$ or $\mathcal{T}$ we write s^I the vector $[s(p_i)]_{i \in I}$ or $[s(., p_i)]_{i \in I}$ (see figure 1).

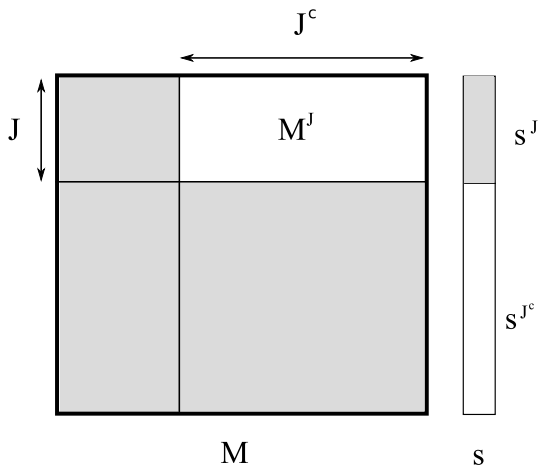

Figure 1: Matrix and vectors associated to a subset J of $\{1, \cdots, k\}$.

Let C a coupler on $\mathcal{P}$. C is said to be **reducible** if there exists $x \in \mathbb{R}^n$ and a non trivial part J of $\{1, \cdots, n\}$ such that $L_c^J(x)$ has non-invertible coefficients (as elements of $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$). C is **non-reducible** if it's not reducible.

The next step consists in "cleaning" the space of periodic solutions in order to exclude some degenerated and/or trivial cases. First the trivial periodic solutions: Any s in $A(\tau, C)$ is said to be non trivial if for all $p \in \mathcal{P}$, neither $s(., p)$ and $c(e)(., p)$ are constant maps.

Now we consider some trajectories that are degenerated in a stronger sense than the one of trivial trajectories: our aim is to avoid periodic trajectories which oscillating behaviors are the only consequences of the coupler's dynamic (we focus on the synchronization of interacting oscillating systems, not on some systems oscillating due to an external force, and we have already mentioned that this kind of external force can be included in the population itself). Such trajectories are those for which there exists a state b satisfying:

$$L_c\big(s(t,.)\big)\big(s(t,.) - b\big) = 0$$

In terms of matrices, we then have for the cell p_i:

$$\sum_{j=1}^{k} c_{ij}\big(s(t,p_i)\big)\big(s(t,p_j) - b(p_j)\big) = 0$$

In other words, the preceding equality shows that the vector $[s(., p_1), \cdots, s(., p_k)]$ is not linearly free, according to the compounds of $L_c\big(e(t,.)\big)$. These observations lead us to the following definition: Let $\mathcal{P}$ be a finite population, C a tissular coupler on $\mathcal{P}$ and τ a period on $\mathcal{P}$. A non-trivial element $s \in A(\tau, C)$

is said to be C**-free** if for any $b \in \mathcal{S}$ and for any non-trivial $J \subset \{1, \cdots, k\}$ we have:

$$L_c\big(s(t,.)\big)^J.\big(s(t,.)-b\big)^{J^c} \neq 0$$

In fact, we can prove that, when C is non-reducible, a non C-free element is degenerated because it comes from a solution of a system on a population which cardinal is strictly lower than k.

Here is the first step to frequencies locking as it deals with dependency of periods [5]:

Theorem 3.1. *If C is non-reducible and if $A(\tau, C)$ contains a non-trivial and C-free element, then $\tau(\mathcal{P})$ is a dependent set.*

Finally we extend this result to a more powerful conclusion. In order to prove synchronization, we must add a last condition on C. This is done introducing the following definition: Let C be a uniformly linear coupler on $\mathcal{P}$. C is said to be strongly non-reducible if it is non-reducible and if for all $x \in \mathbb{R}^n$ and all non trivial subset $J \subset \{1, \ldots, k\}$ with more than $\frac{k}{2}$ elements, $L_c^J(x)$ is injective. **Remark** Because of its dimensions, $L_c^J(x)$ can't be injective if J contains less than $\frac{k}{2}$ elements.

We can now state our main result [5]:

Theorem 3.2. *Let $\mathcal{P}$ be a finite population of cardinal k, C a tissular coupler, uniformly linear and strongly non-reducible and τ a period on $\mathcal{P}$. Suppose $s \in A(\tau, C)$ isn't trivial and is non-reducible. If more than $\frac{k}{2}$ cells have the same period then all cells have the same period.*

4 Synchronization in the case of an infinite population

4.1 Measures associated to an uniformly linear coupling

In the case of an infinite population we need new tools, especially in order to measure the sets related to a period introduced in section 2.2. But this cannot be done independently from the coupler itself. This is why we introduce the following family of measures on $\mathcal{P}$, one for each cell p:

$$\lambda_p(B) = \int_B \|c_d(p,q)\|\, dq$$

We recall that the support of a measure is defined to be the largest closed subset of $\mathcal{P}$ for which every open neighborhood of every cell of the set has positive measure. Let S_p stand for the support of $c_d(p,.)$, this support indicates where the measure λ_p lives and then which cells influence p in its evolution.

In virtue of c_d's continuity, we know that for any measurable subset B of $\mathcal{P}$, the following equivalence holds (with λ the Lebesgue measure):

$$\lambda_p(B) > 0 \Leftrightarrow \lambda(B \cap S_p) > 0$$

Moreover, in the case of a diffuse coupler, this family of measures is uniformly absolutely continuous with respect to λ:

Lemma 4.1. *If C is a uniformly linear and diffuse coupler, then there exists a constant $m_C > 0$ such that for any measurable subset B of $\mathcal{P}$ we have:*

$$\lambda_p(B) \leq m_C \lambda(B), \ \forall p \in \mathcal{P}$$

4.2 Useful subsets of $A(\tau, C)$

The kind of systems we want to handle comes from differential equations studied near limit cycles, so, as in the finite population case, we need to avoid some degenerated trajectories. In order to do so, if $s \in \mathcal{T}^\tau$, we define the set of all cells on which the variations of s are not negligible. Precisely, if $\varepsilon > 0$, we define the following set:

$$V^\varepsilon(s) = \left\{ p \in \mathcal{P} \;\middle|\; \inf_{c \in \mathbb{R}^n} \|s(.,p) - c\| > \varepsilon \right\}$$

We call ε-pseudokernel of C the set:

$$\mathrm{Ker}^\varepsilon(C) = \{ s \in \mathcal{T}^\tau, \quad V^\varepsilon(C(s)) \neq \mathcal{P} \}$$

An element in the ε-kernel of C contains, for at least one cell p, an application $s(.,p)$ which image under C is almost constant. Those solutions are out of interest here because they can be seen as solutions of a system on a sub-population of C. Indeed, for such a p, $s'(.,p)$ is negligible and $s(.,p)$ is nearly constant. In other words, we only consider trajectories whose dynamics in each cell is"truly" periodic.

Given a period τ on $\mathcal{P}$, any element s of $A(\omega, C)$ and $\delta > 0$, the following set gathers cells whiose period are "not isolated", it will be naturally linked to $V^\varepsilon(s)$:

$$R_C^\delta(\tau) = \left\{ p \in \mathcal{P} \;\middle|\; \lambda_p\left(\mathcal{P}_p^{\mathbb{Q}}\right) > \delta \right\}$$

If $p \in R_C^\delta(\tau)$, there is a set, which measure is at least δ, consisting of cells with periods commensurable with $\tau(p)$. Moreover, those cells are situated in the neighborhood of p defined by S_p.

4.3 Ergodic alternative

As in the finite case, we will use this alternative within the following form [5]:

Proposition 4.1. *Let τ be a period on $\mathcal{P}$ and C a tissular diffuse coupler, then for all s in $A(\tau, C)$ we have:*

$$C(s)(t,p) = \int_{\mathcal{P}_p^{\mathbb{Q}}} c_d(s(t,p),p,q)s(t,q)dq + \int_{\mathcal{P}-\mathcal{P}_p^{\mathbb{Q}}} c_d(s(t,p),p,q)\tau(q)\widehat{s}(0,q)dq$$

This results clearly shows that the ergodic theorem acts like a filter on periodic solutions, separating commensurable periods from the others. It is a key element of our proofs.

4.4 Synchronization with uniformly linear diffuse tissular coupler

As it is suggested in the title, in this section the kind of coupler we handle has the following shape:

$$C(s) = \int_{\mathcal{P}} c_d(.,q)s(q)dq = L_C.s$$

Here is the first result explaining the link between the sets V and R. It will be generalized in the next result:

Proposition 4.2. *Let τ be a period on $\mathcal{P}$, for any s in $A(\tau, C)$ the following inclusion holds:*

$$V^0(C(s)) \subset R_C^0(\tau)$$

Noticing that the elements of $A(\tau, C)$ are uniformly bounded on I, we can then consider $\|s\| = \sup_{I\times\mathcal{P}} \|s(t,p)\|$ and use it to acquire the generalization:

Proposition 4.3. *Let τ be a period on $\mathcal{P}$, for any s in $A(\tau, C)$ we have:*

$$V^{\varepsilon}(C(s)) \subset R_C^{\frac{\varepsilon}{\|s\|}}(\tau)$$

In particular, as $\mathcal{P}$ is a compact set, we can state:

Corollary 4.1. *Let τ be a period on $\mathcal{P}$ and suppose there exists $\varepsilon > 0$ and $e \in A(\tau, C) - \mathrm{Ker}^{\varepsilon}(C)$, then there exists a finite number of cells $p_1, \ldots, p_j$ such that:*

$$\mathcal{P} = \bigcup_{i=1}^{j} \mathcal{P}_{p_i}^{\mathbb{Q}}$$

The latter result leads us to this somewhat different result:

Corollary 4.2. *Let τ be a period on $\mathcal{P}$ and suppose there exists $\varepsilon > 0$ and $e \in A(\tau, C) - \mathrm{Ker}^{\varepsilon}(C)$, then there exists a sequence $\{p_i\}_{i\in\mathbb{N}}$ such that:*

$$\mathcal{P}_{p_i}^{\mathbb{Q}} = \bigcup_i \mathcal{P}_{p_i}^1$$

These results indicate a great reduction of the problem as they show that the periods cannot be arbitrary scattered. We are now able to prove the main result of this section. In order to handle only true cases of synchronization, we add an assumption on τ so that we won't have to deal with sub-period stuff:

Definition 4.1. *τ is said to be simple if* $2\,\mathrm{Conv}(\tau(\mathcal{P})) \cap \mathrm{Conv}(\tau(\mathcal{P})) = \emptyset$.

Under this hypothesis, we have the following global synchronization result:

Theorem 4.1. *Suppose τ is a simple period on $\mathcal{P}$, if there exists $\varepsilon > 0$ and s in $A(\tau, C) - \mathrm{Ker}^{\varepsilon}(C)$, then τ is a constant map on $\mathcal{P}$.*

Remark: To prove this result, we used the continuity of τ, deduced from the one of c_d. Nevertheless, leaving this assumption aside, we can get to an interesting result which could lead to generalization. We only need corollary 4.2 (we point that its validity does not directly depends on c_d's continuity but on the uniform absolute continuity of the family λ_p and on the fact that $\|c_d(.,.)\|$ is uniformly bounded on $\mathcal{P}^2$).

Definition 4.2. *The set of point with isolated period, written $\mathcal{P}^0$, is defined as:*

$$\mathcal{P}^0 = \left\{p \in \mathcal{P}, \quad \lambda_p\left(\mathcal{P}_p^1\right) = 0\right\}$$

The following proposition shows that under minimal assumptions, there are almost no points with isolated period:

Theorem 4.2. *Suppose there exists a partition as in the corollary 4.2 and that*

$$\mathcal{P} \subset \bigcup_{p\in\mathcal{P}} S_p$$

then $\lambda(\mathcal{P}_0) = 0$.

The latter result gives a global information that is verified in a general case: almost no cell has an isolated period. Those two results are really different, the first one is less theoretical but refers to stronger hypotheses than the second one, which comes under the field of measure theory.

4.5 Synchronization with uniformly linear tissular coupler

Now, we consider the general case, leaving aside the assumption that C is diffuse. Thus, in this last and short section, we consider a uniformly linear tissular coupler on $\mathcal{P}$ (c_a needs no more to be zero). Like in the finite population case, the first step consists in detecting and eliminating degenerated solutions:

Definition 4.3. *Let $s \in \mathcal{T}(I, \mathcal{E}_b)$ and $\varepsilon \geq 0$, s is said to be an ε-eigenvector (for C) if:*

$$C(s) - c_a \in \mathrm{Ker}^{\varepsilon}(C)$$

An ε-eigenvector is therefore a s which contains a map $e(., p)$ uniformly near a hypothetic eigenvector of C for the eigenvalue $c_a(p)$.

Finally, here is the generalization of theorem 4.1:

Theorem 4.3. *Let τ be a simple period on $\mathcal{P}$ and $\varepsilon > 0$. If there exists $s \in A(\tau, C)$ which is not an ε-eigenvector, then τ is a constant map.*

5 Conclusion

In this work we have built a general framework, the tissular coupling, to handle a wide variety of coupled systems, and therefore a wide class of complex systems. We focused on an emergent property of those dynamical systems: synchronization, and precisely frequencies locking. We used the notion of tissular coupling to show that the synchronization issue may be addressed differently. Usually one observes solutions of particular coupled systems and shows that within suitable conditions synchronization must occur. Those results are often qualitatively dependent of the studied systems and do not stand in the general case. We tried to change our point of view and to bring completing results. These are less precise than usual ones as we don't prove that synchronization ultimately happens. Instead, we consider the problem at its end: if one supposes that coupled systems oscillate, then they must be synchronized. The loss in time evolution informations is compensated by very general results, almost independent from the individual differential systems to be coupled. We believe that this complementary approach of frequencies locking will lead to future developments as it brings about many unused mathematical tools. Other examples of the use of tissular coupling can be found in [5] on emergence of spatial patterns. All those work and results are seem to prove that tissular coupling will be a prolific framework, completing what already exists.

6 References

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